NPL National Physical Laboratory | Centre for Carbon Measurement

Understanding the Law of Propagation of Uncertainties

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esa

fiducial reference measurements for satellite ocean colour

CEOS

Metrology for Earth Observation and Climate

http://www.emceoc.org

EMRP
European Metrology Research Programme
Programme of EURAMET

The EMRP is jointly funded by the EMRP participating countries within EURAMET and the European Union

EURAMET
European Association of National Metrology Institutes

The Law of Propagation of Uncertainties

$$u_c^2(y) = \boxed{\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i)} + \boxed{2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i, x_j)}$$

Things you might already "know"

$$u_c^2(y) = \boxed{\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i)} + \boxed{2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i, x_j)}$$

Has a sensitivity coefficient
Adding in quadrature (% or units)

This term is to do with correlation

Averages reduce by $1/\sqrt{n}$

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Building on that starting point

$$u_c^2(y) = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i, x_j)$$

- The Law of Propagation of Uncertainties

Has a sensitivity coefficient

Adding in quadrature (% or units)

- How many readings you should average

Averages reduce by $1/\sqrt{n}$

- What correlation is

This term is to do with correlation

At the end of this module, you should understand

- The Law of Propagation of Uncertainties
  - Why the sensitivity coefficient is central to the law
  - How to determine the sensitivity coefficients
  - What ‘adding in quadrature’ really means

- How many readings you should average
  - What the uncertainty associated with an average is
  - How you know you have enough readings
  - When you’ve taken too many readings

- What correlation is
  - What the uncertainties are for averaging partially correlated data
  - How to estimate covariance from numerical and experimental data
  - How to deal with not knowing what the correlation is

$$u_{\mathrm{c}}^{2}(y) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \right)^2 u^2(x_i) + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} u(x_i, x_j)$$

THE LAW OF PROPAGATION OF UNCERTAINTIES

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At the end of this module, you should understand

- The Law of Propagation of Uncertainties
  - Why the sensitivity coefficient is central to the law
  - How to determine the sensitivity coefficients
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$$u_{\mathrm{c}}^{2}(y)=\sum_{i=1}^{n}\boxed{\left(\frac{\partial f}{\partial x_i}\right)}^{2}u^{2}(x_i)+2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i,x_j)$$

Sensitivity Coefficients

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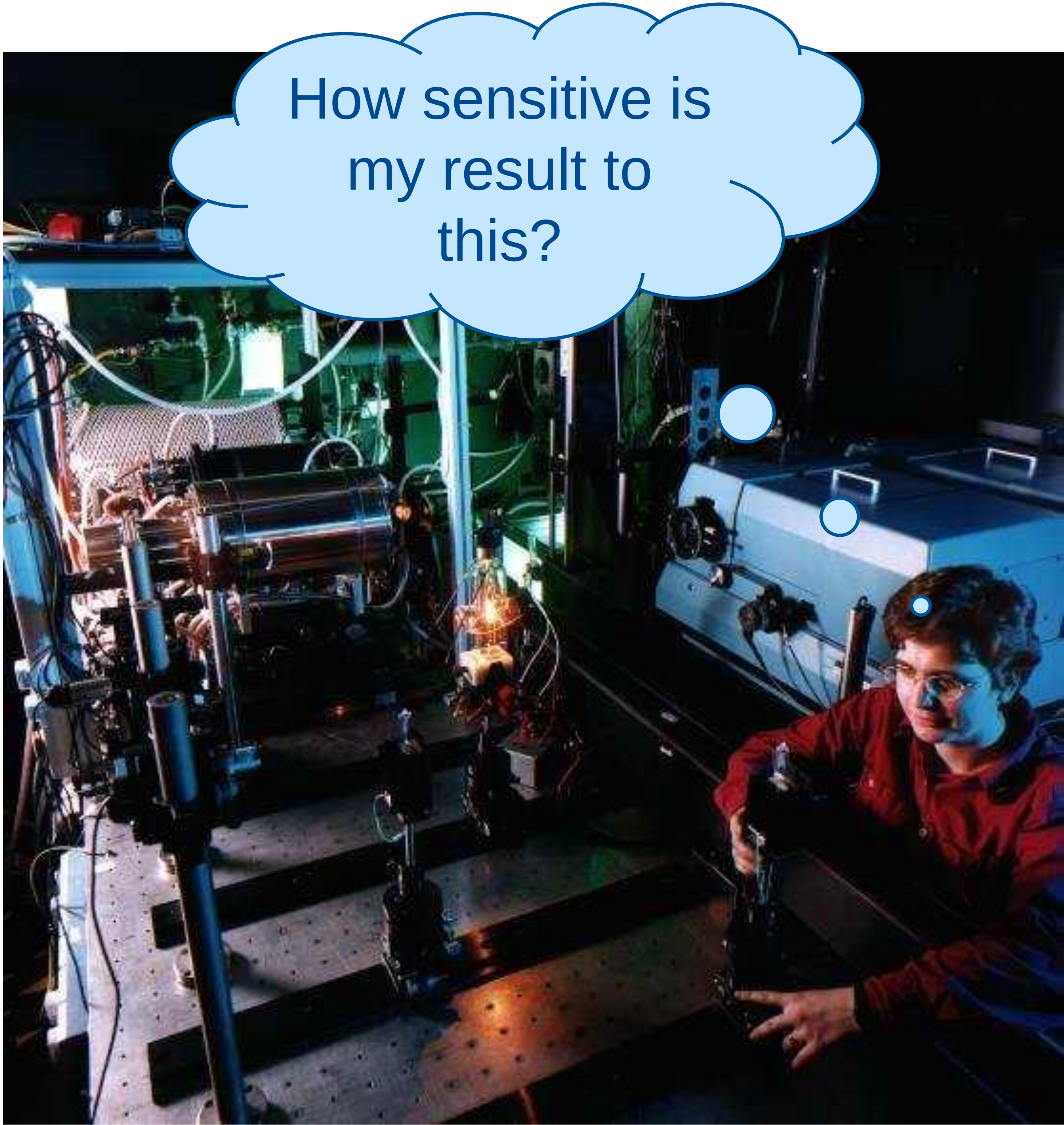
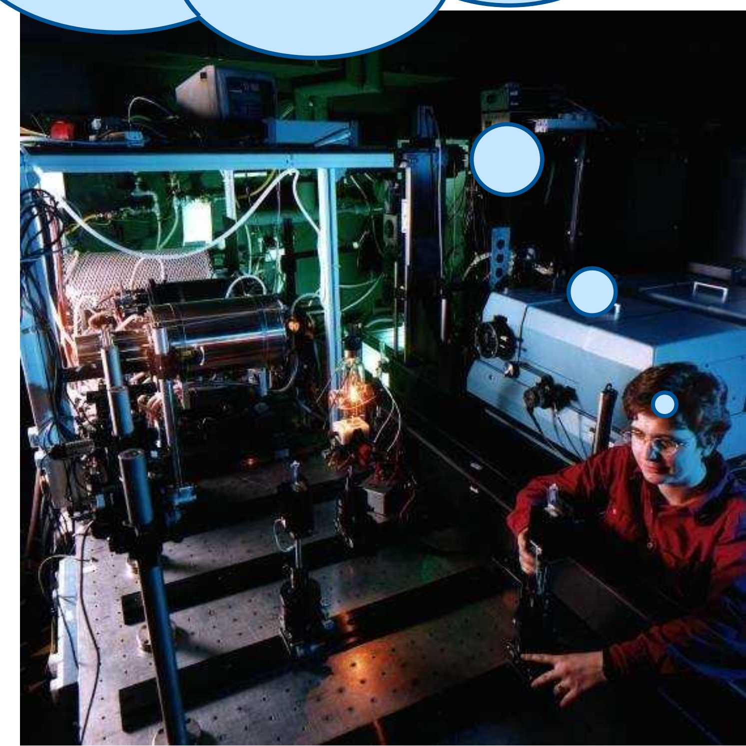
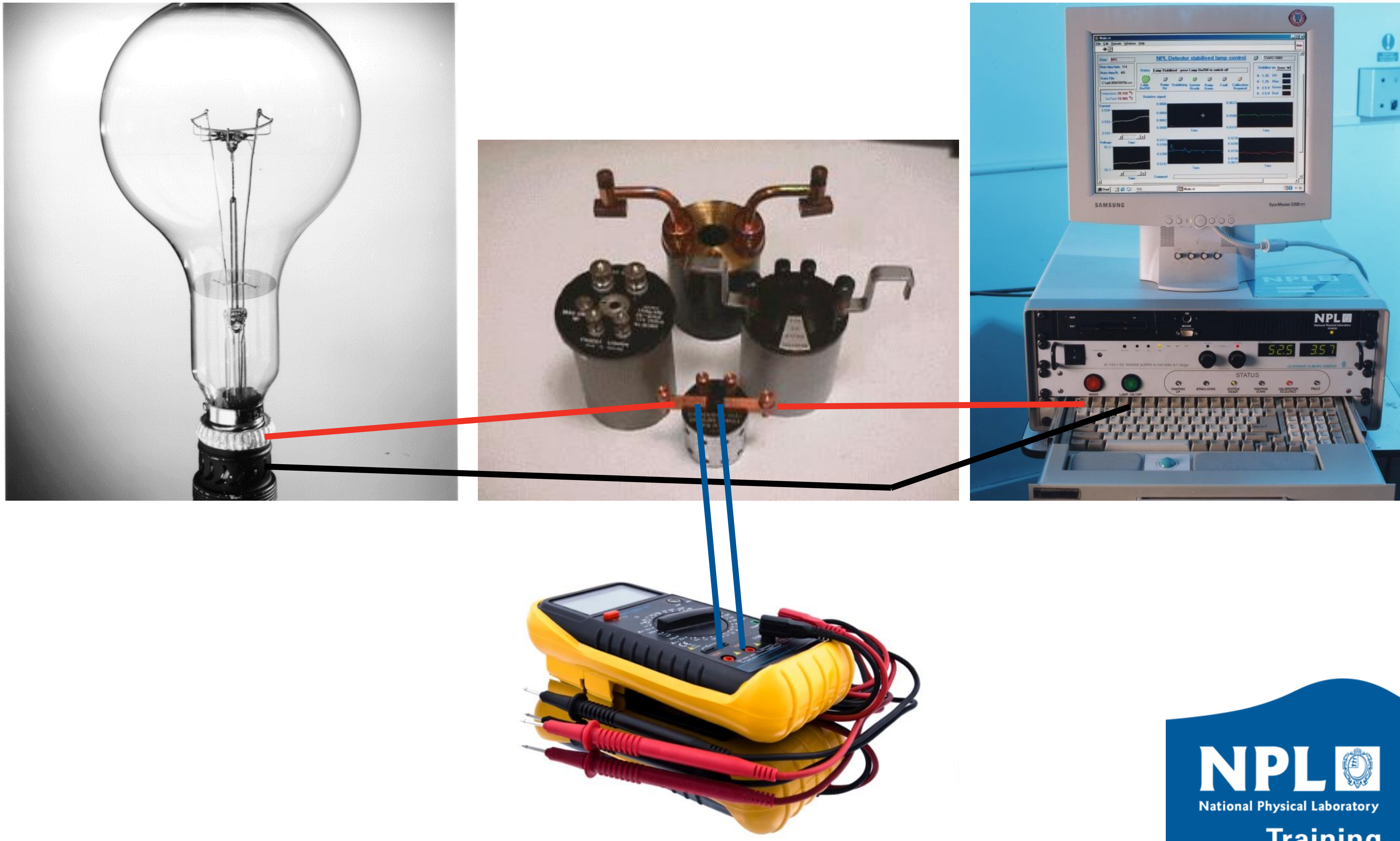

How sensitive is my result to this?

Do an experiment to find out!

How sensitive is my result to this?



Lamp power supply:
What is the sensitivity of the irradiance to the lamp current?



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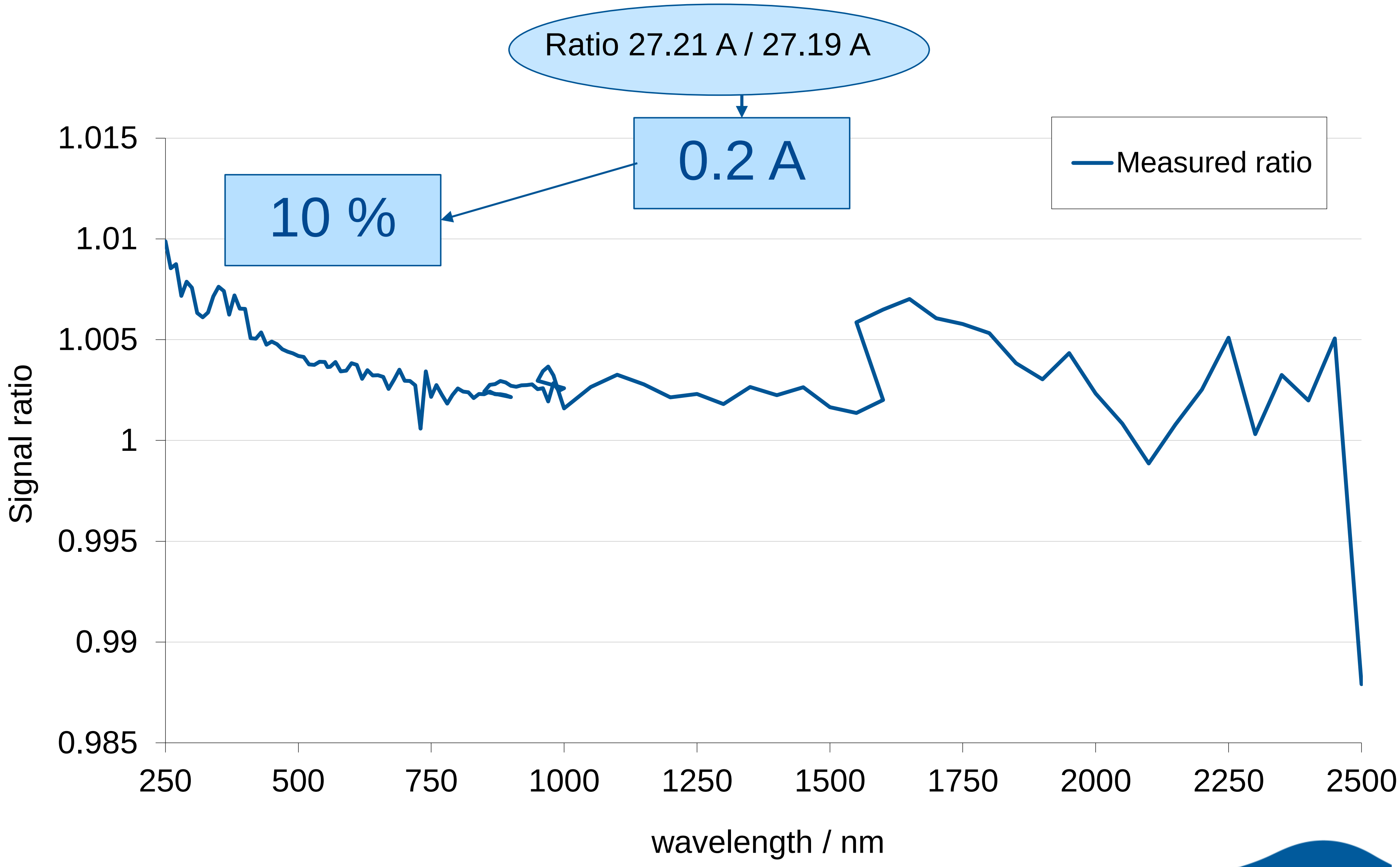

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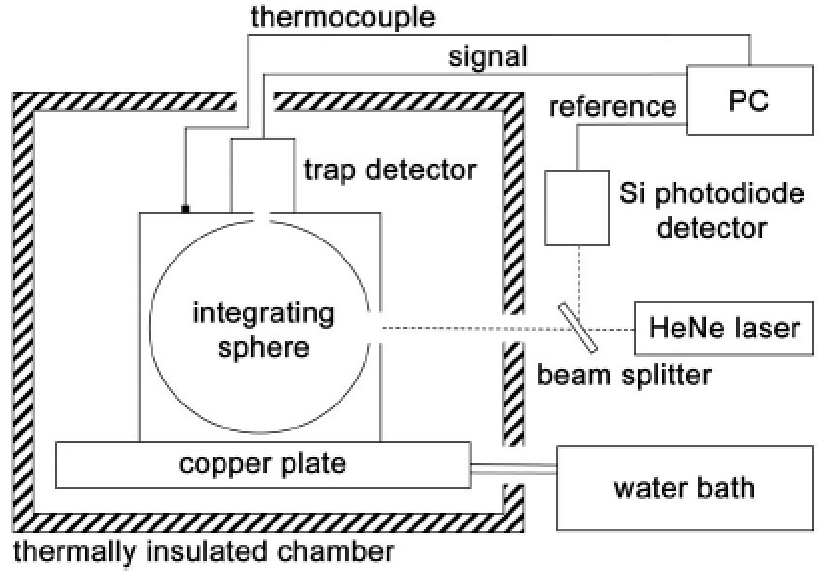
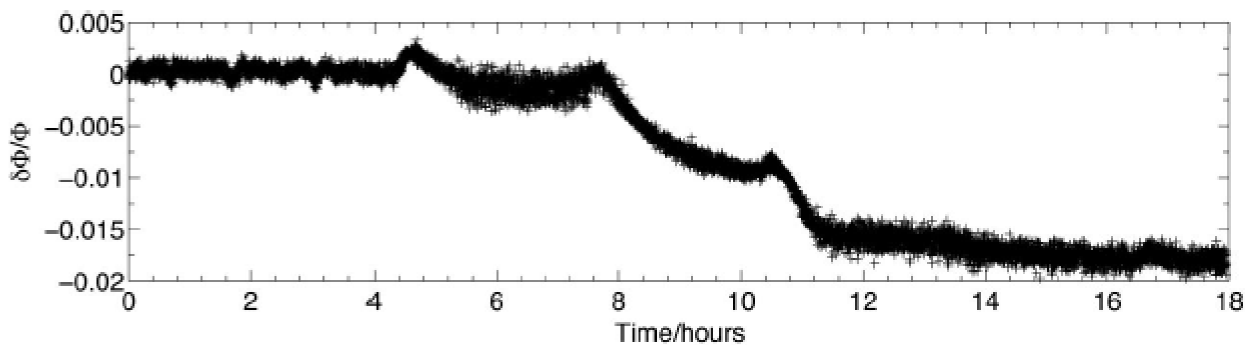
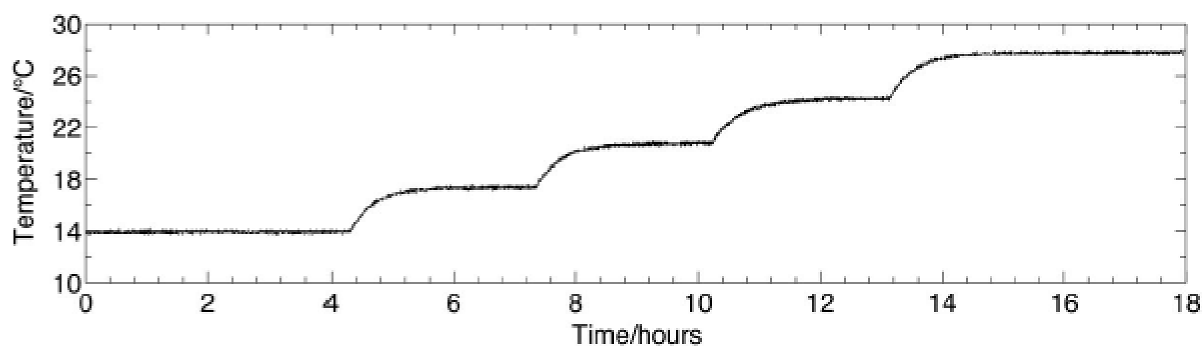
PTFE Sphere phase transition



1.5 %

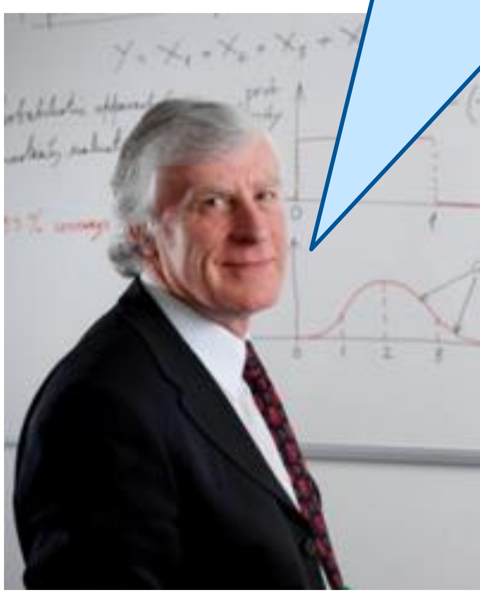
6 °C

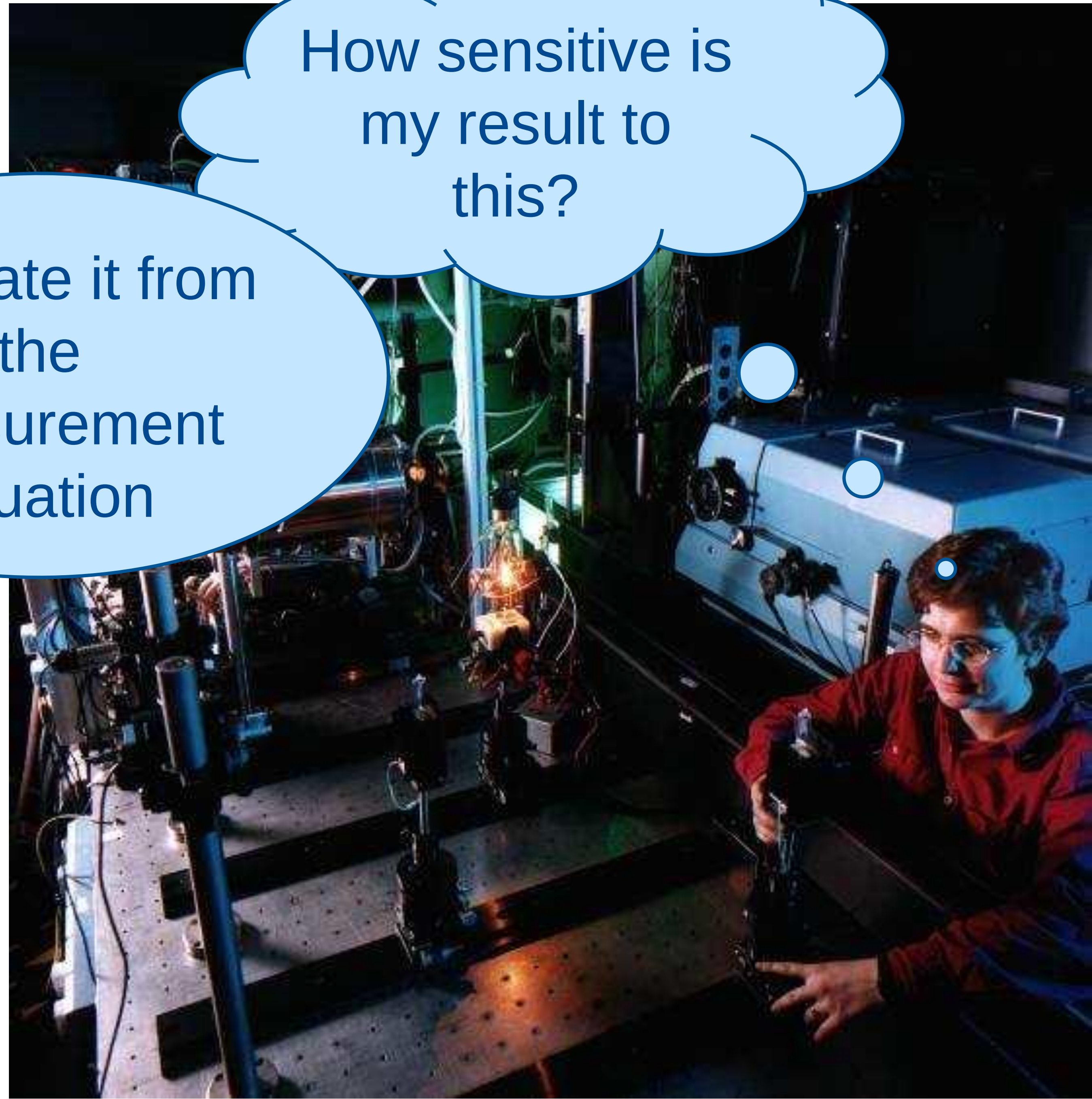




How sensitive is my result to this?

Calculate it from the measurement equation

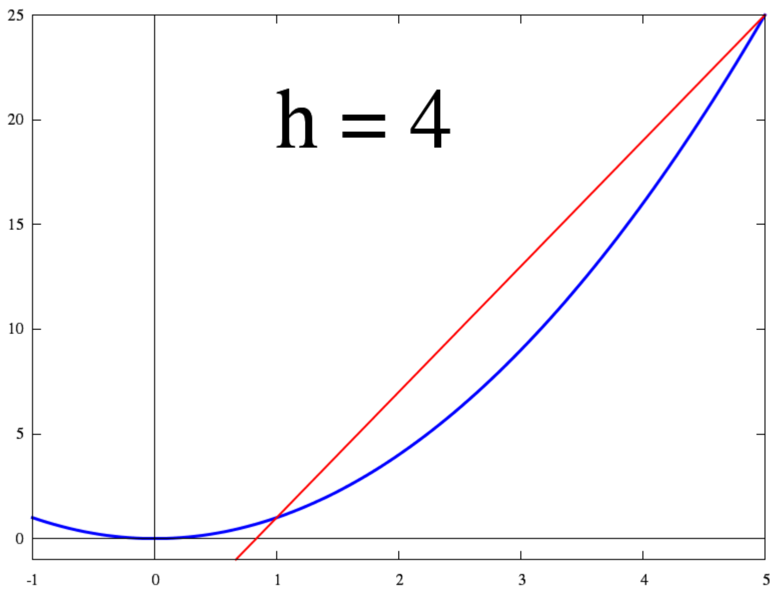




Analytical sensitivity coefficients

$$y = f\left(x_1, x_2, x_3, \ldots\right)$$

$$c_{x_i} = \frac{\partial f}{\partial x_i}$$



The local slope:

Translating a change in one parameter to a change in the other

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When it works well

$$V_{\mathrm{s}} = V_{\mathrm{light}} - V_{\mathrm{dark}}$$

$$\frac{\partial V_{\mathrm{s}}}{\partial V_{\mathrm{light}}} = 1 \qquad \frac{\partial V_{\mathrm{s}}}{\partial V_{\mathrm{dark}}} = -1$$

Very simple case

When it works well

$$L_{\mathrm{s}} = \frac{E_{\mathrm{FEL}}\beta_{0-45}}{\pi}\frac{d_{\mathrm{cal}}^{2}}{d_{\mathrm{use}}^{2}}$$

$$\frac{\partial L_{\mathrm{s}}}{\partial d_{\mathrm{use}}} = -2\frac{E_{\mathrm{FEL}}\beta_{0-45}}{\pi}\frac{d_{\mathrm{cal}}^{2}}{d_{\mathrm{use}}^{3}} = \frac{-2L_{\mathrm{s}}}{d_{\mathrm{use}}}$$

Inverse square law leads to a 2

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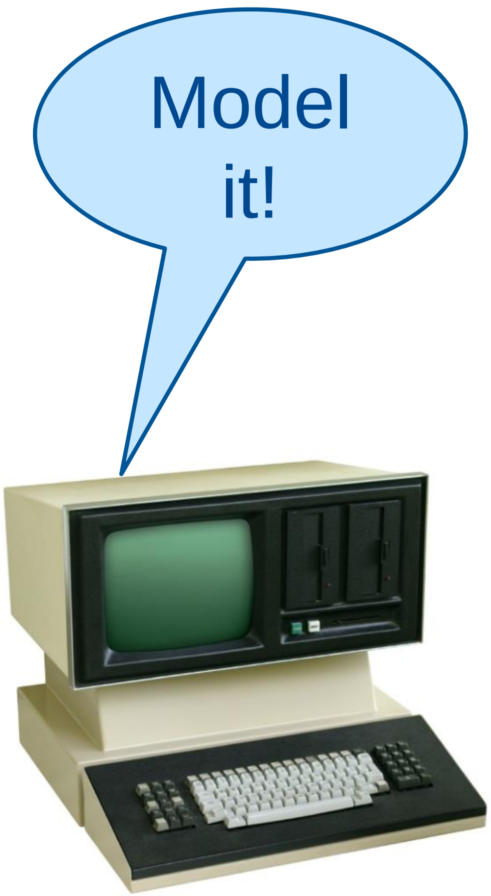
When it doesn’t work well

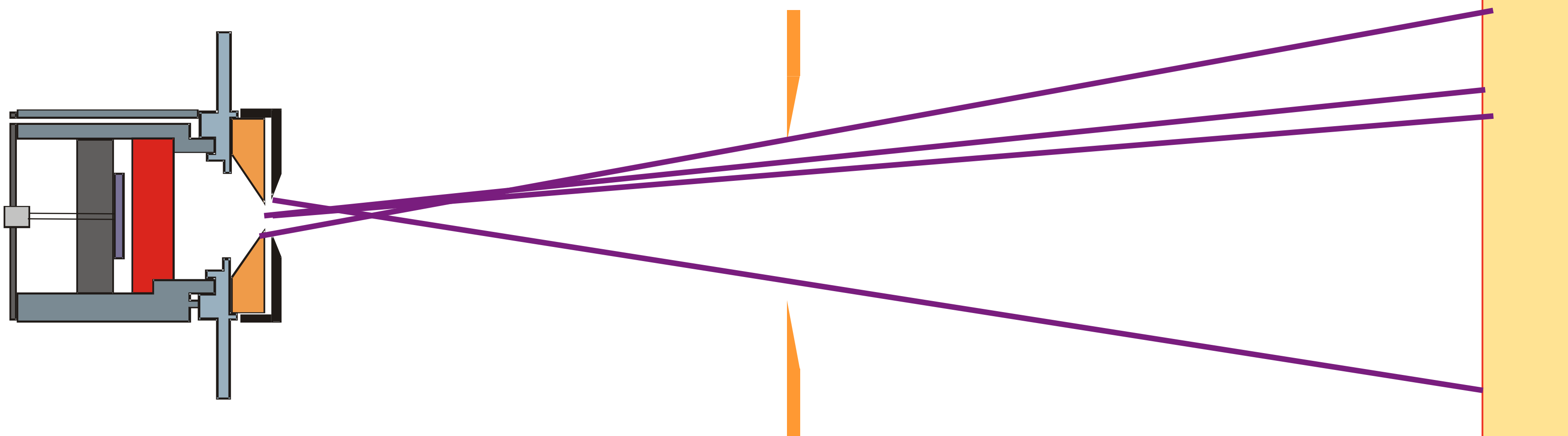
- Because you can’t write a relationship
- Because it’s too difficult to differentiate
- Because it’s a program not an equation


Model it!


How sensitive is my result to this?

Radiance through double aperture system

$$\Phi_{A_1} = g\pi L$$



$$g_{\ell\text{-d}} = \frac{2\pi r_1^2 r_2^2}{\left(r_1^2 + r_2^2 + x^2\right) + \sqrt{\left(r_1^2 + r_2^2 + x^2\right)^2 - 4r_1^2 r_2^2}}$$

By differentiating …

$$g_{\ell\text{-d}} = \frac{2\pi r_1^2 r_2^2}{\left(r_1^2 + r_2^2 + x^2\right) + \sqrt{\left(r_1^2 + r_2^2 + x^2\right)^2 - 4r_1^2 r_2^2}} \qquad g = 2\pi r_1^2 r_2^2 \alpha^{-1}$$

$$\beta = \sqrt{\left(r_1^2 + r_2^2 + d^2\right)^2 - 4r_1^2 r_2^2}$$

$$\alpha = \left(r_1^2 + r_2^2 + d^2\right) + \beta$$

$$\frac{\partial g}{\partial r_1} = \frac{4\pi r_1 r_2^2}{\alpha}\left[1 - \frac{r_1^2}{\alpha}\left(\frac{\alpha - 2r_2^2}{\beta}\right)\right]$$

$$\frac{\partial g}{\partial r_2} = \frac{4\pi r_1^2 r_2}{\alpha}\left[1 - \frac{r_2^2}{\alpha}\left(\frac{\alpha - 2r_1^2}{\beta}\right)\right]$$

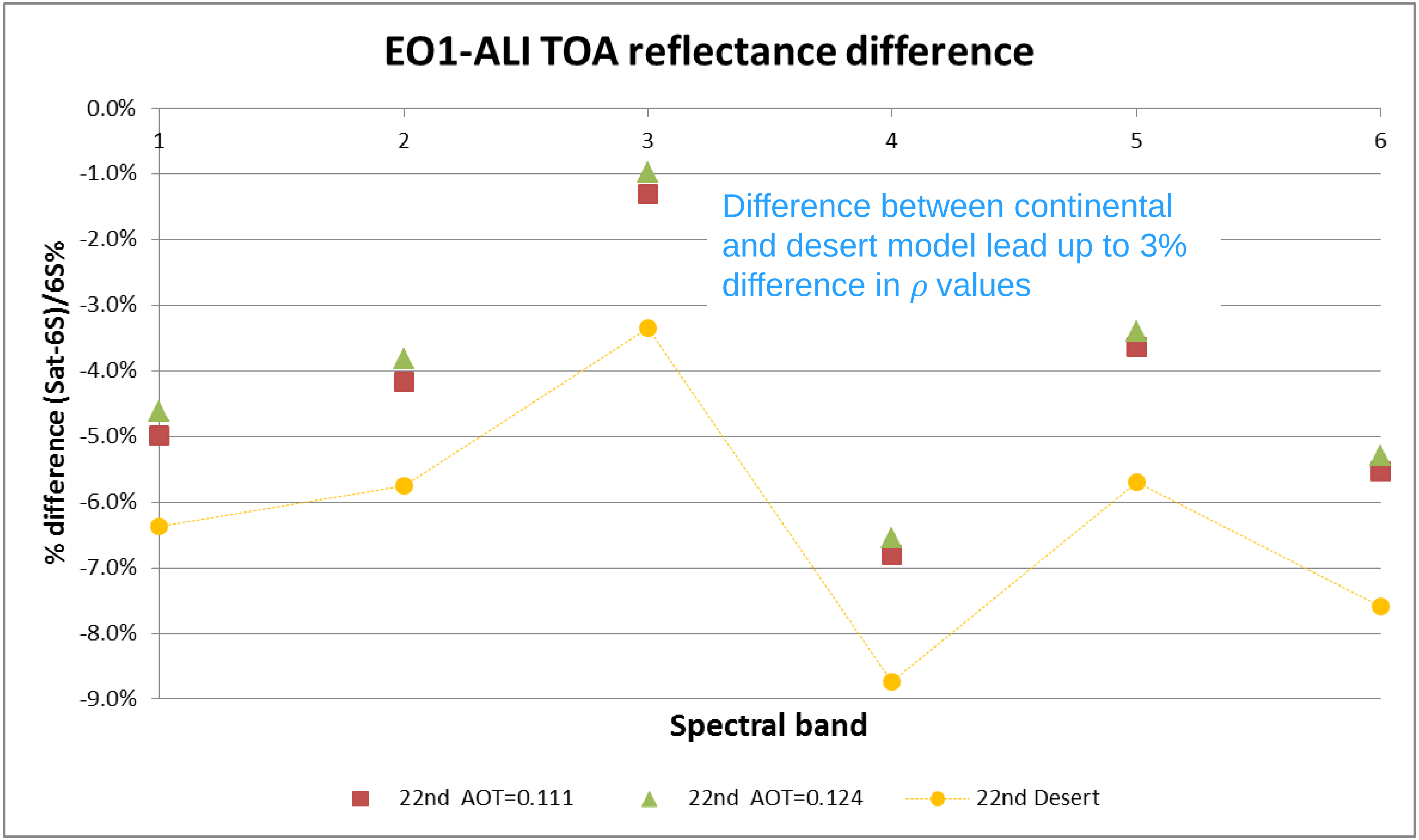
$$\frac{\partial g}{\partial d} = \frac{-4\pi r_1^2 r_2^2 d}{\alpha\beta}$$

Or by “modelling”…

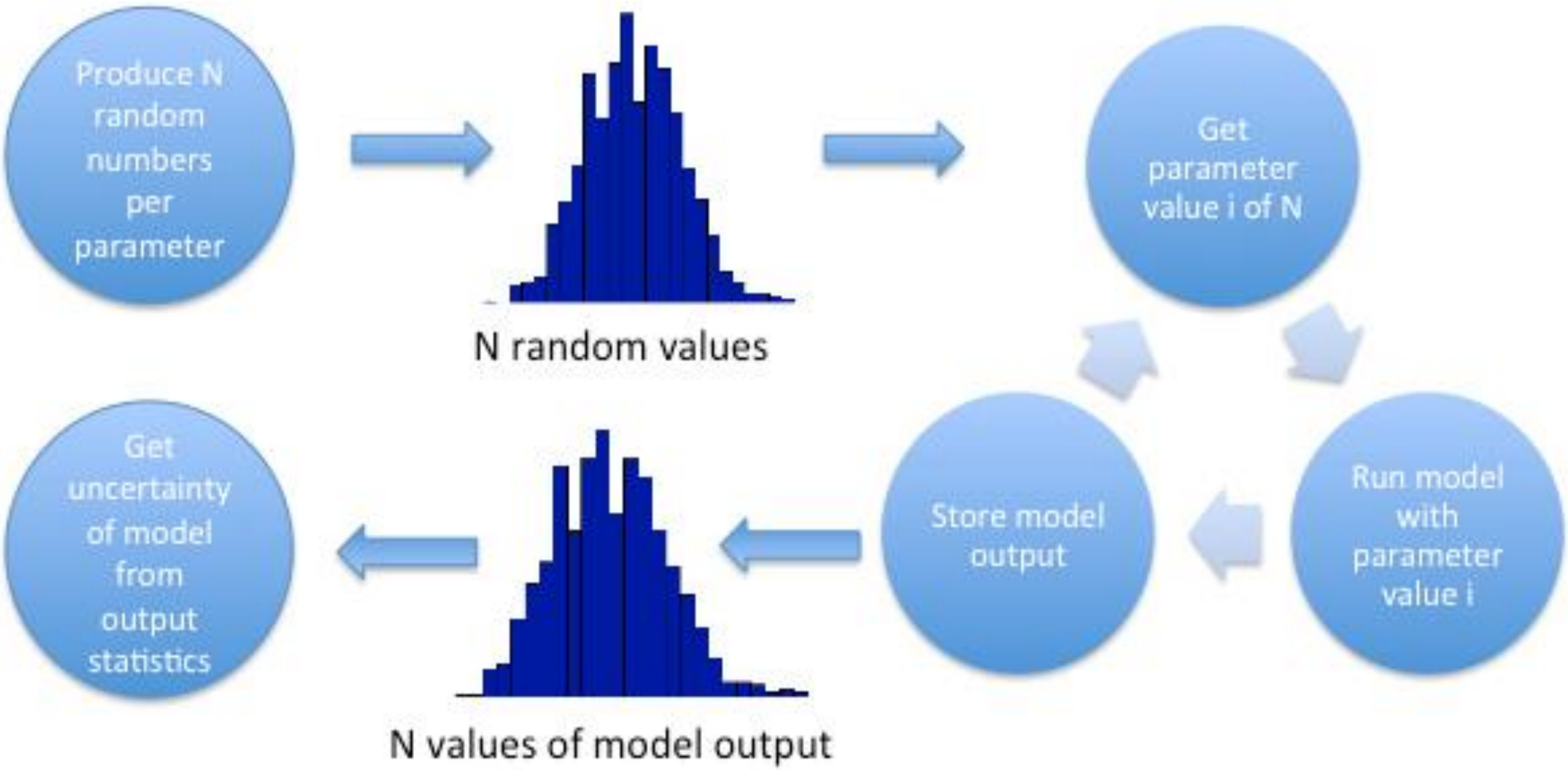
	A	B	C	D	E	F	G
4				Uncertainty in diameter or distance			
5	**This first section calculates g for the straightforward case**			microns	metres		3.1E-10
6	radius of first aperture / m	0.0025	5	1	1E-06		57.986%
7	radius of second aperture / m	0.0015	3	1	1E-06		
8	distance between apertures / m	0.3	300	40	0.00004		
9							
10	sum of squares	0.0900085			**Uncertainty associated with g**		
11	4 r1^2 r2^2	5.625E-11			due to first aperture		0.0400%
12	square root	0.0900085	beta		due to second aperture		0.0667%
13	bottom line	0.180017	alpha		due to distance		-0.0267%
14	g	4.908E-10			***Combined uncertainty associated with g***		**0.082%**
15							
16	**This section recalculates g, changing one parameter at a time**						
17		Changing first aperture	Changing second aperture	Changing distance			
18	radius of first aperture / m	0.0025005	0.0025	0.0025			
19	radius of second aperture / m	0.0015	0.0015005	0.0015			
20	distance between apertures / m	0.3	0.3	0.30004			
21							
22	sum of squares	0.090008503	0.090008502	0.090032502			
23	4 r1^2 r2^2	5.62725E-11	5.62875E-11	5.625E-11			
24	square root	0.090008502	0.090008501	0.090032501			
25	bottom line	0.180017005	0.180017003	0.180065003			
26	g	4.910E-10	4.912E-10	4.907E-10			
27		0.0400%	0.0667%	-0.0267%			
28							
29							

Sensitivity to atmospheric model choice

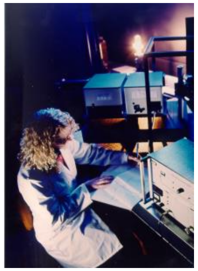
Tuz Gölü CEOS comparison 2010



Monte Carlo simulation



Sensitivity Coefficients

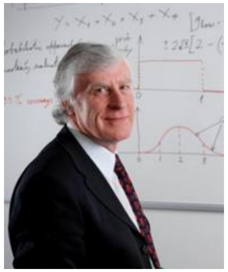


Experimentally

Vary in the lab and see what happens

Numerically

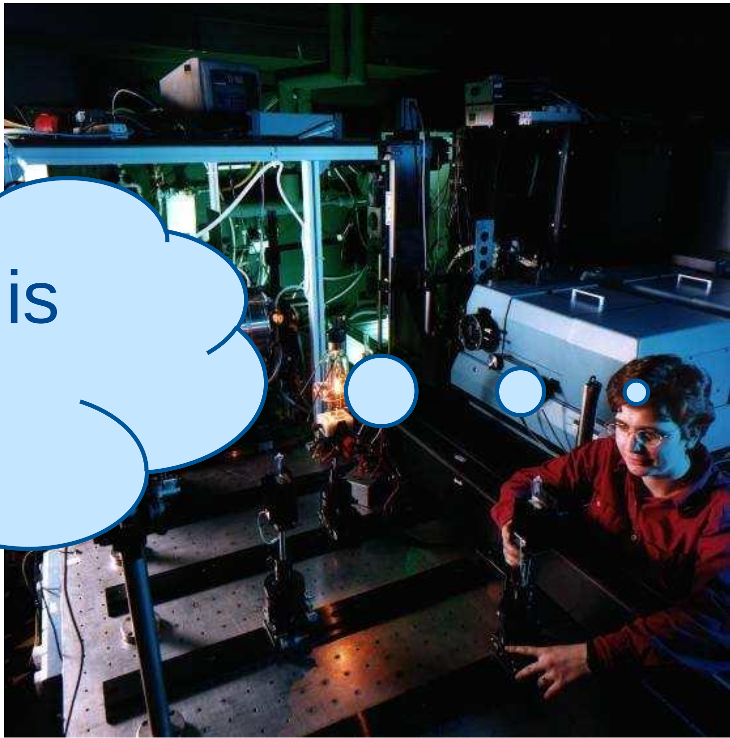
Vary in the model and see what happens



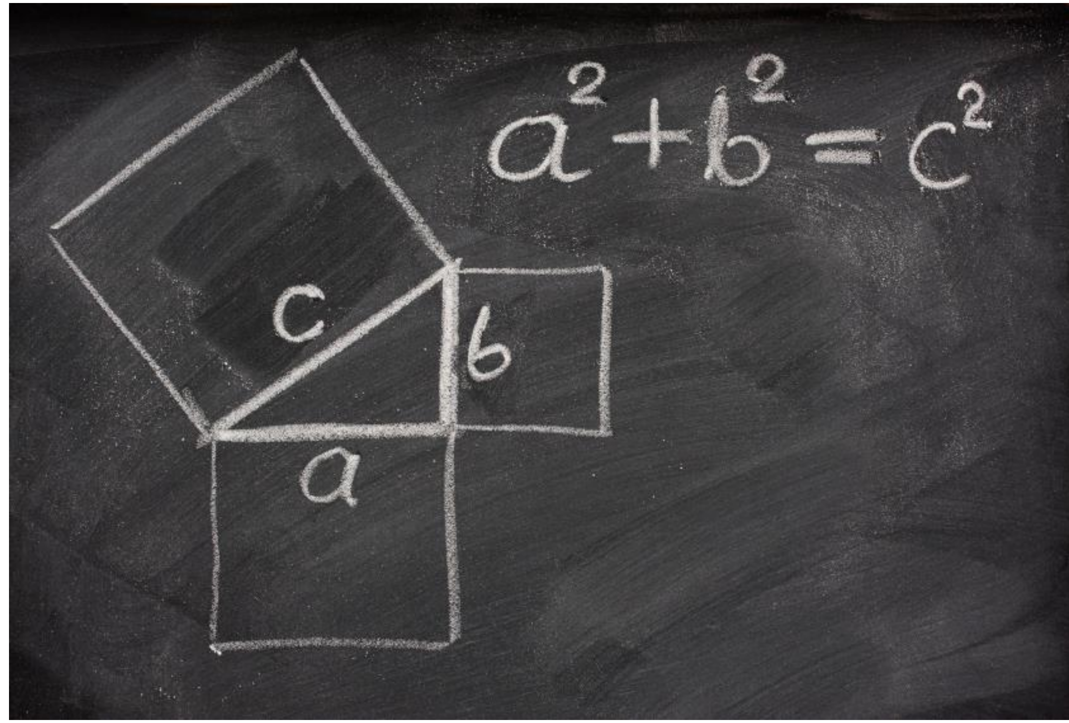
Mathematically

$$c_{x_i} = \frac{\partial f}{\partial x_i}$$

How sensitive is my result to this?



All are acceptable



Adding in Quadrature

$$u_c^2(y)=\boxed{\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i)}+2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i,x_j)$$

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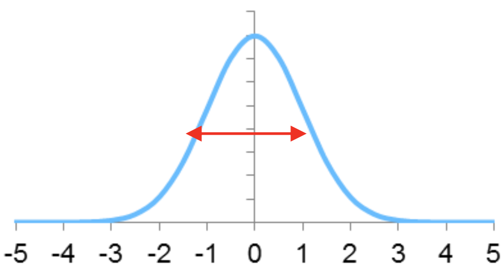
Why quadrature?

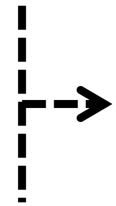
Uncertainty



Describes the spread

Combining distributions leads to the quadrature rule

Error



Difference to the (unknowable) true value

It's very unlikely that all will be maximum in same direction!

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Uncertainty expression

Relative uncertainty:

5 mW m^{-2} nm^{-1} $\pm$ 0.2 %

i.e. uncertainty expressed as a percentage

Absolute uncertainty:

5 mW m^{-2} nm^{-1} $\pm$ 0.01 mW m^{-2} nm^{-1} i.e. uncertainty expressed in the native measurement units

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Absolute uncertainty:

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Adding in quadrature for absolute models

$$y = A + B + C$$

$$\frac{\partial y}{\partial A} = \frac{\partial y}{\partial B} = \frac{\partial y}{\partial C} = 1$$

$$u^2(y) = u^2(A) + u^2(B) + u^2(C)$$

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Adding in quadrature for absolute models

$$y = A - B$$

$$\frac{\partial y}{\partial A} = 1 \qquad \frac{\partial y}{\partial B} = -1$$

$$u^2(y) = u^2(A) + (-1)^2 u^2(B)$$

$$u^2(y) = u^2(A) + u^2(B)$$

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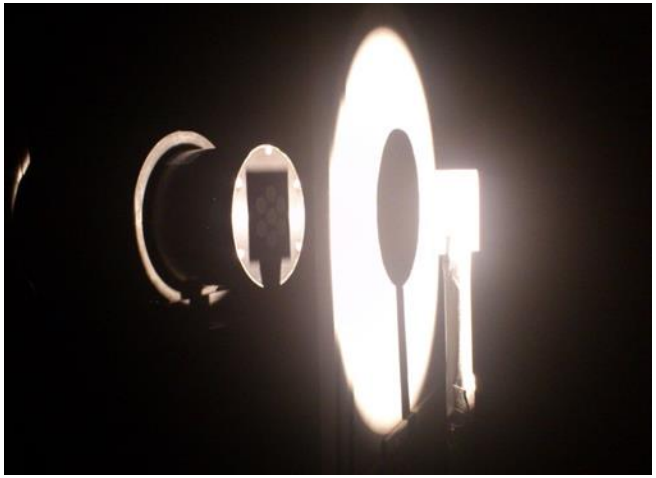
What behaves like an absolute model?

Distance = Measured + aperture thickness

Signal = Light signal – Dark signal



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Uncertainty expression

Relative uncertainty:

5 mW m^{-2} nm^{-1} $\pm$ 0.2 %

i.e. uncertainty expressed as a percentage

Absolute uncertainty:

5 mW m^{-2} nm^{-1} $\pm$ 0.01 mW m^{-2} nm^{-1} i.e. uncertainty expressed in the native measurement units

Adding in quadrature for relative models

$$y = A \times B$$

$$\frac{\partial y}{\partial A} = B = \frac{y}{A}$$

$$\frac{\partial y}{\partial B} = A = \frac{y}{B}$$

$$u^2(y) = \left(\frac{y}{A}\right)^2 u^2(A) + \left(\frac{y}{B}\right)^2 u^2(B)$$

$$\left(\frac{u(y)}{y}\right)^2 = \left(\frac{u(A)}{A}\right)^2 + \left(\frac{u(B)}{B}\right)^2$$

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Adding in quadrature for relative models

$$y = A/B$$

$$\frac{\partial y}{\partial A} = \frac{1}{B} = \frac{y}{A}$$

$$\frac{\partial y}{\partial B} = \frac{-A}{B^2} = \frac{-y}{B}$$

$$u^2(y) = \left(\frac{y}{A}\right)^2 u^2(A) + \left(\frac{-y}{B}\right)^2 u^2(B)$$

$$\left(\frac{u(y)}{y}\right)^2 = \left(\frac{u(A)}{A}\right)^2 + \left(\frac{u(B)}{B}\right)^2$$

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Adding in quadrature for relative models

$$y = A/B^2$$

$$\frac{\partial y}{\partial A} = \frac{1}{B^2} = \frac{y}{A}$$

$$\frac{\partial y}{\partial B} = \frac{-2A}{B^3} = \frac{-2y}{B}$$

$$u^2(y) = \left(\frac{y}{A}\right)^2 u^2(A) + \left(\frac{-2y}{B}\right)^2 u^2(B)$$

$$\left(\frac{u(y)}{y}\right)^2 = \left(\frac{u(A)}{A}\right)^2 + \left(\frac{2u(B)}{B}\right)^2$$

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What behaves like a relative model?

- Most radiometric equations!
- Anything where uncertainties are in %

Gain = Radiance / Signal

$$L_s = \frac{E_{\mathrm{FEL}}\,\beta_{0°:45°}}{\pi}\,\frac{d_{\mathrm{cal}}^2}{d_{\mathrm{use}}^2}$$

$$\frac{\partial L_s}{\partial E_{\mathrm{FEL}}} = \frac{L_s}{E_{\mathrm{FEL}}}$$

$$\frac{\partial L_s}{\partial \beta_{0°:45°}} = \frac{L_s}{\beta_{0°:45°}}$$

$$\frac{\partial L_s}{\partial d_{\mathrm{use}}} = \frac{-2L_s}{d_{\mathrm{use}}}$$

$$\left(\frac{u\left(L_s\right)}{L_s}\right)^2 = \left(\frac{u\left(E_{\mathrm{FEL}}\right)}{E_{\mathrm{FEL}}}\right)^2 + \left(\frac{u\left(\beta_{0°:45°}\right)}{\beta_{0°:45°}}\right)^2 + \left(\frac{2u\left(d_{\mathrm{use}}\right)}{d_{\mathrm{use}}}\right)^2$$

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At the end of this module, you should understand

- The Law of Propagation of Uncertainties
  - Why the sensitivity coefficient is central to the law
  - How to determine the sensitivity coefficients
  - What ‘adding in quadrature’ really means

- How many readings you should average
  - What the uncertainty associated with an average is
  - How you know you have enough readings
  - When you’ve taken too many readings

- What correlation is
  - What the uncertainties are for averaging partially correlated data
  - How to estimate covariance from numerical and experimental data
  - How to deal with not knowing what the correlation is

HOW MANY READINGS SHOULD BE AVERAGED?

At the end of this module, you should understand

- The Law of Propagation of Uncertainties
  - Why the sensitivity coefficient is central to the law
  - How to determine the sensitivity coefficients
  - What ‘adding in quadrature’ really means
  - What the ‘other term’ – the correlation bit means

- How many readings you should average
  - What the uncertainty associated with an average is
  - How you know you have enough readings
  - When you’ve taken too many readings

- What correlation is
  - What the uncertainties are for averaging partially correlated data
  - How to estimate covariance from numerical and experimental data
  - How to deal with not knowing what the correlation is

Applying the Law of Propagation of Uncertainties to an average

Averaging three readings

$$E_{\mathrm{M}} = \frac{E_1 + E_2 + E_3}{3}$$

$$\frac{\partial E_{\mathrm{M}}}{\partial E_1} = \frac{\partial E_{\mathrm{M}}}{\partial E_2} = \frac{\partial E_{\mathrm{M}}}{\partial E_3} = \frac{1}{3}$$

$$u^2(E_{\mathrm{M}}) = \left(\frac{1}{3}\right)^2 u^2(E_1) + \left(\frac{1}{3}\right)^2 u^2(E_2) + \left(\frac{1}{3}\right)^2 u^2(E_3)$$

$$u(E_1) = u(E_2) = u(E_3) = u(E_i)$$

$$u^2(E_{\mathrm{M}}) = 3\left(\frac{1}{3}\right)^2 u^2(E_i) \qquad u^2(E_{\mathrm{M}}) = \left(\frac{3}{3^2}\right) u^2(E_i)$$

$$u^2(E_{\mathrm{M}}) = \left(\frac{1}{3}\right) u^2(E_i) \qquad u^2(E_{\mathrm{M}}) = \left(\frac{u(E_i)}{\sqrt{3}}\right)^2 \qquad u(E_{\mathrm{M}}) = \left(\frac{u(E_i)}{\sqrt{3}}\right)$$

Applying the Law of Propagation of Uncertainties to an average

Averaging n readings

$$E_{\mathrm{M}} = \left(\sum_{i=1}^{n} E_i\right) \Big/ n$$

$$\frac{\partial E_{\mathrm{M}}}{\partial E_i} = \frac{1}{n}$$

$$u^2\left(E_{\mathrm{M}}\right) = \left(\sum_{i=1}^{n} u^2\left(E_i\right)\right) \Big/ n$$

$$u^2\left(E_{\mathrm{M}}\right) = n\left(\frac{1}{n}\right)^2 u^2\left(E_i\right) \qquad u^2\left(E_{\mathrm{M}}\right) = \left(\frac{1}{n}\right) u^2\left(E_i\right)$$

$$u^2\left(E_{\mathrm{M}}\right) = \left(\frac{u\left(E_i\right)}{\sqrt{n}}\right)^2 \qquad u\left(E_{\mathrm{M}}\right) = \left(\frac{u\left(E_i\right)}{\sqrt{n}}\right)$$

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Average light – average dark

$$V_{\mathrm{s}} = \frac{1}{N}\sum_{i=1}^{N} V_{\mathrm{light},i} - \frac{1}{M}\sum_{i=1}^{M} V_{\mathrm{dark},j}$$

$$c_{V_{\mathrm{light},i}} = \frac{\partial V}{\partial V_{\mathrm{light},i}} = \frac{1}{N}, \quad i = 1,\ldots,\mathrm{N}$$

$$c_{V_{\mathrm{dark},j}} = \frac{\partial V}{\partial V_{\mathrm{dark},j}} = \frac{-1}{M}, \quad j = 1,\ldots,\mathrm{M}$$

$$u_V^2 = \sum_{i=1}^{N}\left(\frac{1}{N}\right)^2 u_{V_{\mathrm{light},i}}^2 + \sum_{j=1}^{M}\left(\frac{-1}{M}\right)^2 u_{V_{\mathrm{dark},j}}^2$$

$$u_V^2 = \left(\frac{u_{V_{\mathrm{light}}}}{\sqrt{N}}\right)^2 + \left(\frac{u_{V_{\mathrm{dark}}}}{\sqrt{M}}\right)^2$$

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Taking enough readings

Applying the Law of Propagation of Uncertainties to an average

Averaging three readings

$$E_{\mathrm{M}} = \frac{E_1 + E_2 + E_3}{3}$$

$$\frac{\partial E_{\mathrm{M}}}{\partial E_1} = \frac{\partial E_{\mathrm{M}}}{\partial E_2} = \frac{\partial E_{\mathrm{M}}}{\partial E_3} = \frac{1}{3}$$

$$u^2\left(E_{\mathrm{M}}\right) = \left(\frac{1}{3}\right)^2 u^2\left(E_1\right) + \left(\frac{1}{3}\right)^2 u^2\left(E_2\right) + \left(\frac{1}{3}\right)^2 u^2\left(E_3\right)$$

$$\boxed{u\left(E_1\right) = u\left(E_2\right) = u\left(E_3\right) = u\left(E_i\right)}$$

$$u^2\left(E_{\mathrm{M}}\right) = 3\left(\frac{1}{3}\right)^2 u^2\left(E_i\right)$$

$$u^2\left(E_{\mathrm{M}}\right) = \left(\frac{3}{3^2}\right) u^2\left(E_i\right)$$

$$u^2\left(E_{\mathrm{M}}\right) = \left(\frac{1}{3}\right) u^2\left(E_i\right)$$

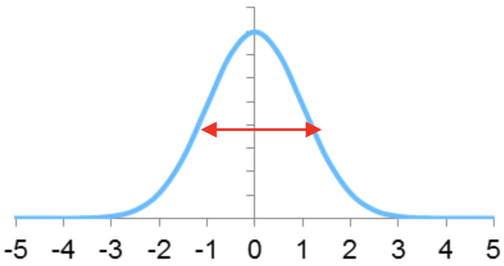
$$u^2\left(E_{\mathrm{M}}\right) = \left(\frac{u\left(E_i\right)}{\sqrt{3}}\right)^2$$

$$u\left(E_{\mathrm{M}}\right) = \left(\frac{u\left(E_i\right)}{\sqrt{3}}\right)$$

What is the uncertainty associated with each reading?

$$u(E_1) = u(E_2) = u(E_3) = u(E_i)$$

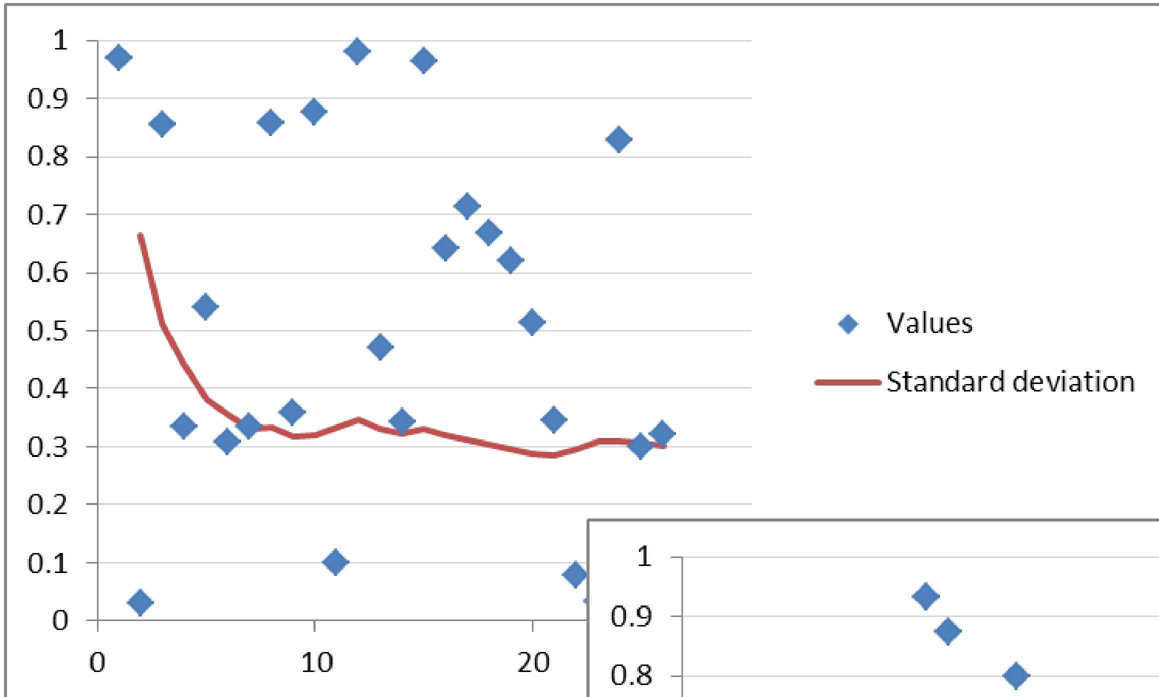
Uncertainty

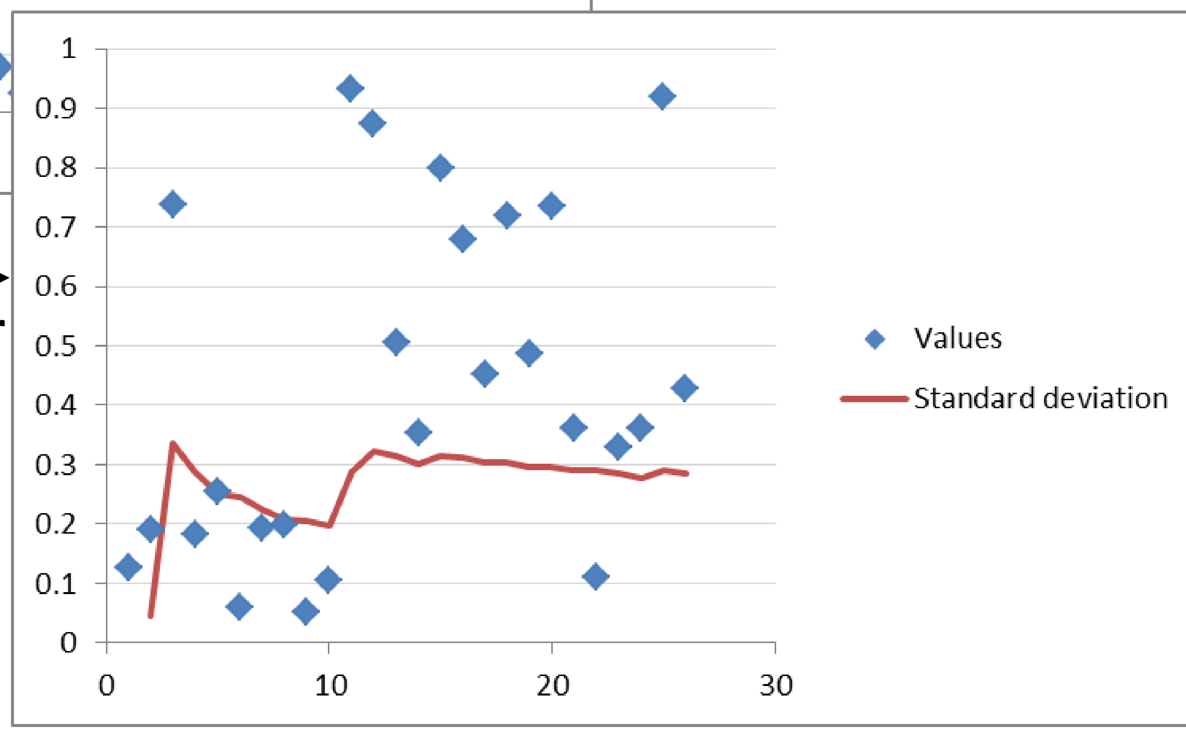


Describes the spread

Standard deviation?

The dangers of taking a standard deviation of a small number of readings

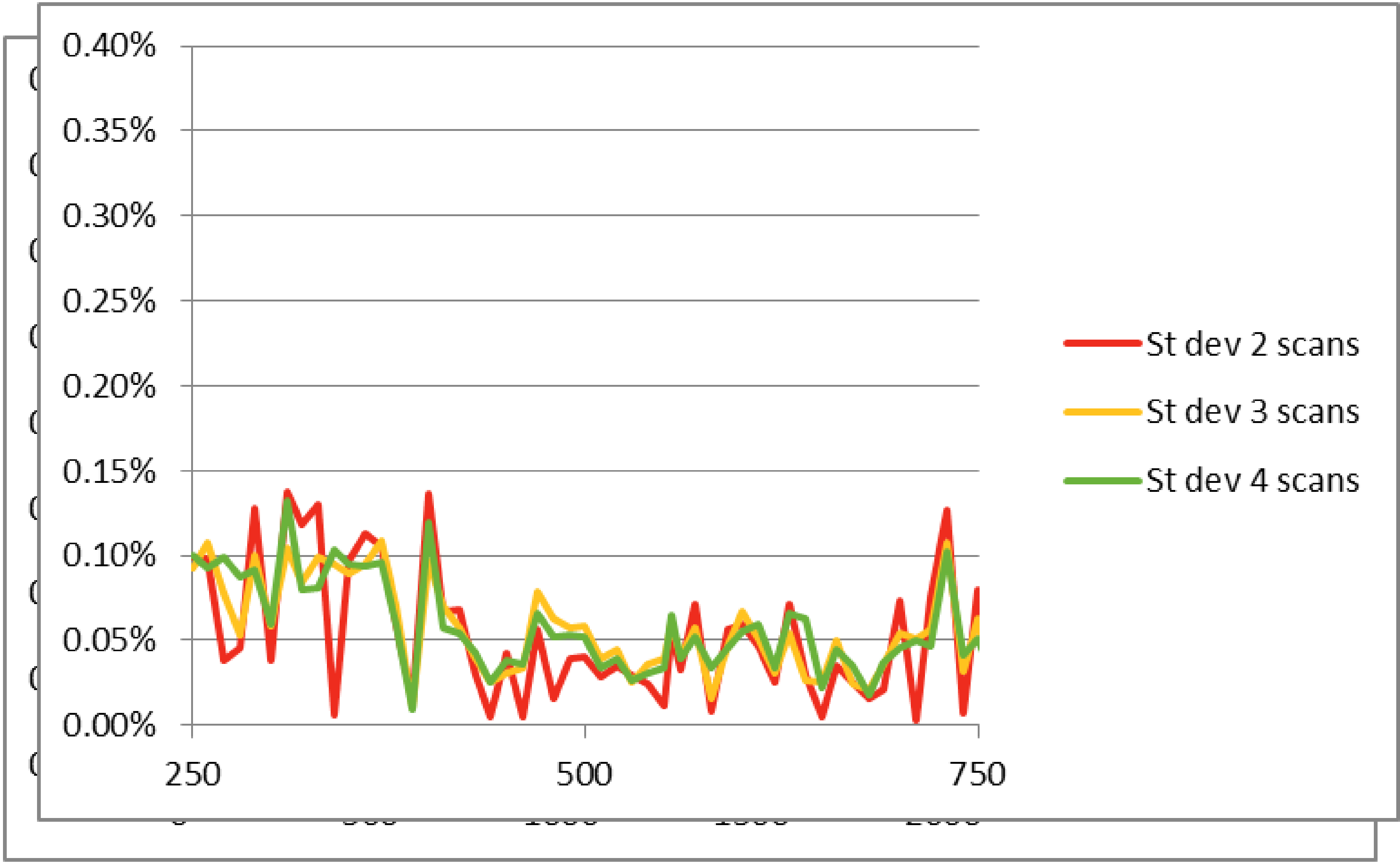




Values and cumulative St dev

Reading number

The dangers of taking a standard deviation from a small number of readings



What you should do

Use more readings!

- Take more measurements today

$$u\left(E_{\mathrm{M}}\right)=\left(\frac{u\left(E_i\right)}{\sqrt{n}}\right)$$

Better estimate of uncertainty

More to average

What you should do

Use more readings!

- Have a facility commissioning phase where you take more measurements

$$u(E_{\mathrm{M}}) = \left(\frac{u(E_i)}{\sqrt{n}} \right)$$

Better estimate of uncertainty

Today's!

From standard deviation of 10: $u(E_i)$

Today two readings, $\sqrt{n} = \sqrt{2}$

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What you should do

Use more readings!

- Compare data from different days

Even if there is a drift:
e.g Use standard deviation of the difference
between two readings

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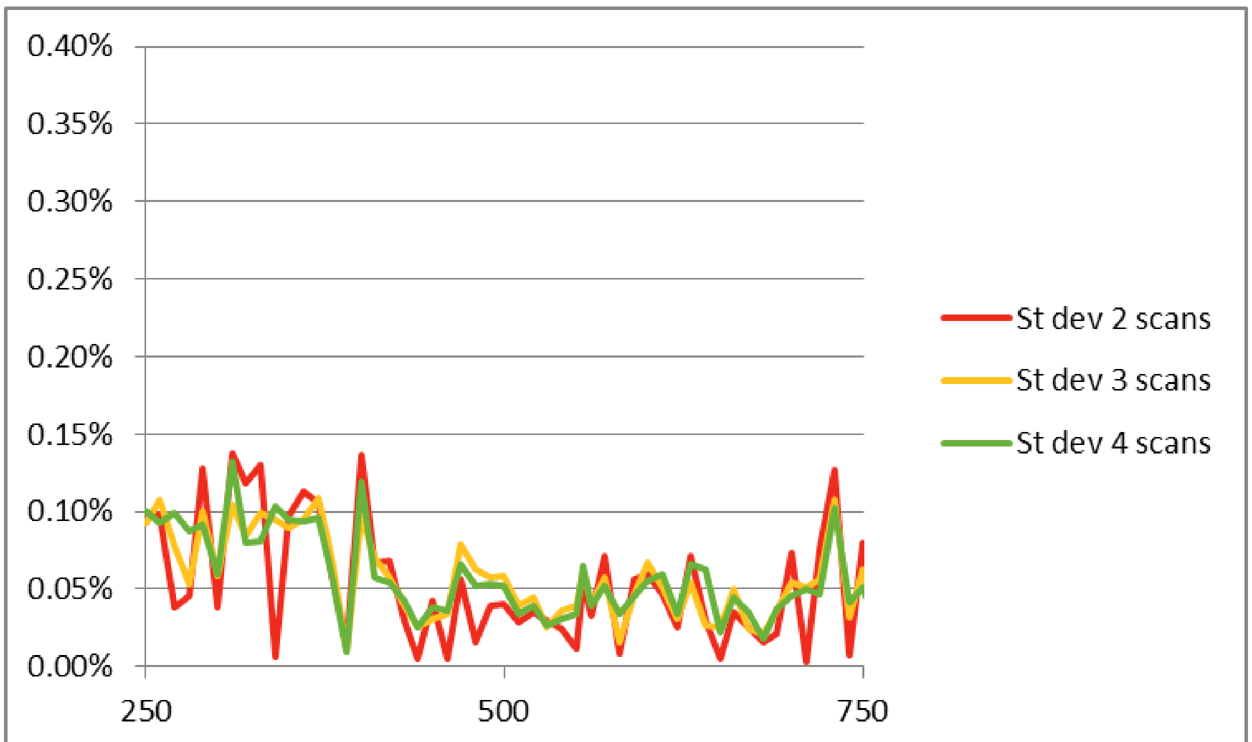
What you should do

Use more readings!

- Compare data from different wavelengths (smooth out the bumps)



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What you should do

Increase the estimate

- Increase standard uncertainty

Will be in revised GUM

$$u_{\text{light,mean}}^{2}=\frac{N-1}{N-3}\left(\frac{s_{\text{light}}}{\sqrt{N}}\right)^{2}$$

What you should do

Use more readings!

- Take more measurements today
- Have a facility commissioning phase
- Compare data from different days
- Compare data from different wavelengths

Increase the estimate

- Increased standard uncertainty

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Use more readings!

But only when it's worth it!

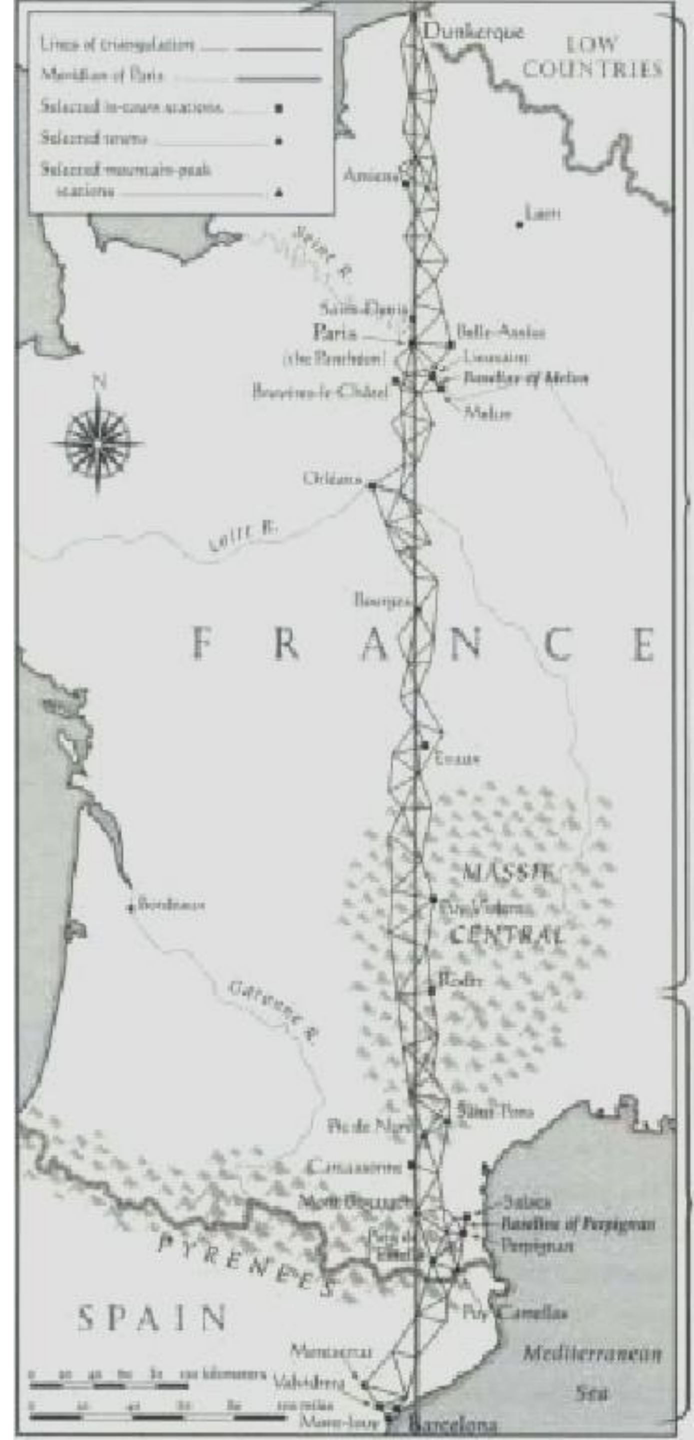
KNOWING WHEN TO STOP!

The metre: 1791 - 1799

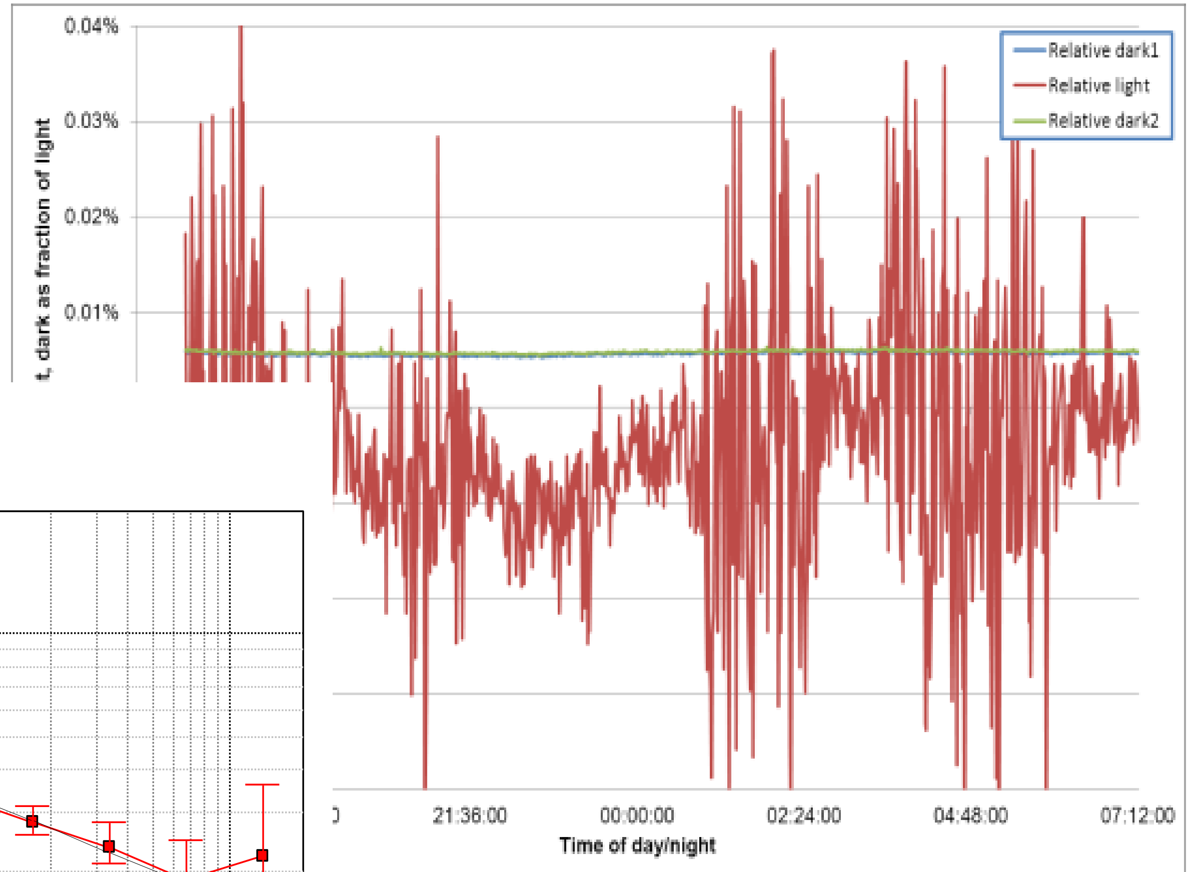
Pierre-Françoise-André Méchain

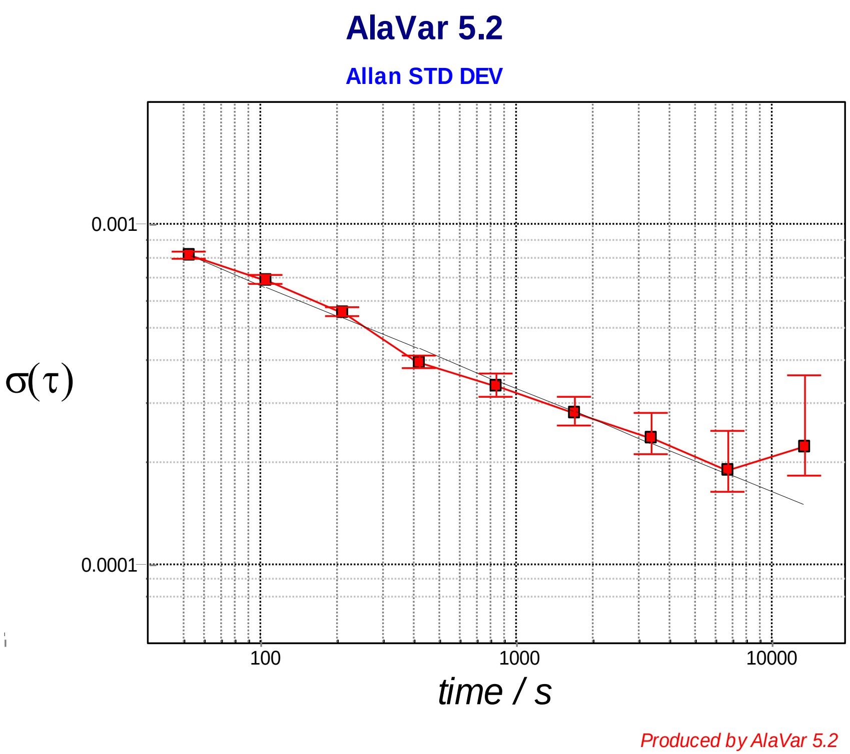
Jean-Baptiste-Joseph Delambre

The Measure of all Things - Ken Alder

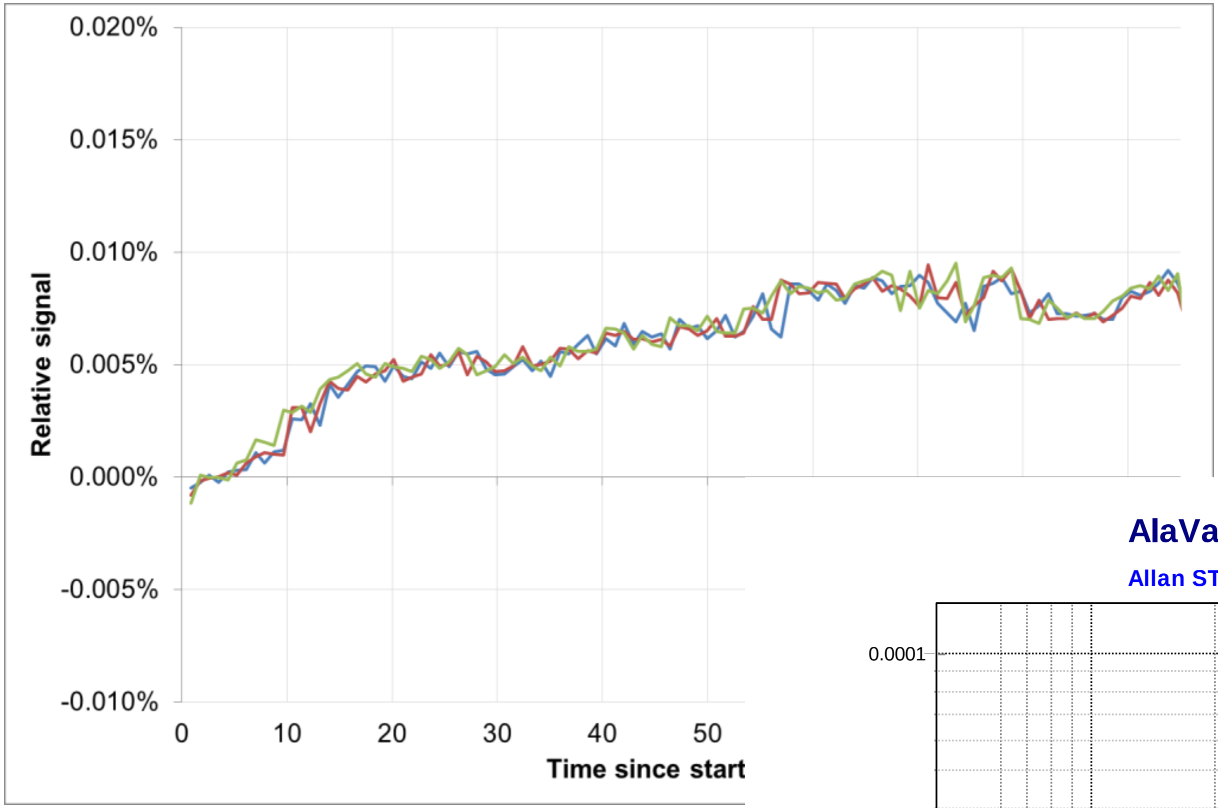


The uncertainties don't keep dropping forever

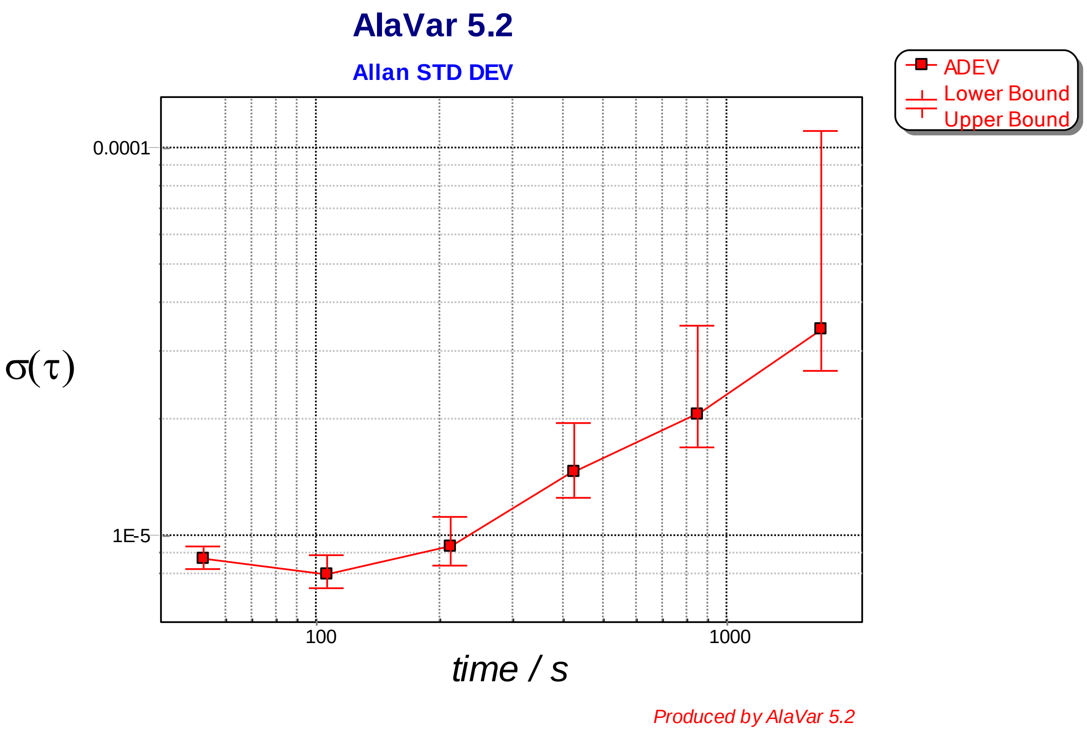




The Allan Deviation – software



www.alamath.com



At the end of this module, you should understand

- The Law of Propagation of Uncertainties
  - Why the sensitivity coefficient is central to the law
  - How to determine the sensitivity coefficients
  - What ‘adding in quadrature’ really means

- How many readings you should average
  - What the uncertainty associated with an average is
  - How you know you have enough readings
  - When you’ve taken too many readings

- What correlation is
  - What the uncertainties are for averaging partially correlated data
  - How to estimate covariance from numerical and experimental data
  - How to deal with not knowing what the correlation is

CORRELATION

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Training

At the end of this module, you should understand

- The Law of Propagation of Uncertainties
  - Why the sensitivity coefficient is central to the law
  - How to determine the sensitivity coefficients
  - What ‘adding in quadrature’ really means

- How many readings you should average
  - What the uncertainty associated with an average is
  - How you know you have enough readings
  - When you’ve taken too many readings

- What correlation is
  - What the uncertainties are for averaging partially correlated data
  - How to estimate covariance from numerical and experimental data
  - How to deal with not knowing what the correlation is

The Law of Propagation of Uncertainties

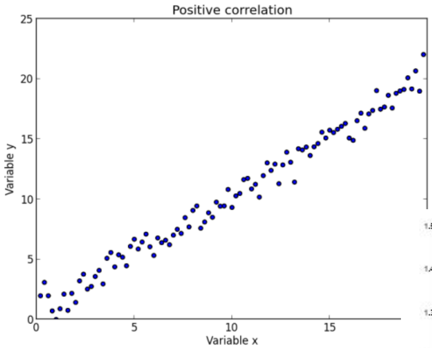
$$u_c^2(y) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \right)^2 u^2(x_i) + \boxed{2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} u(x_i, x_j)}$$

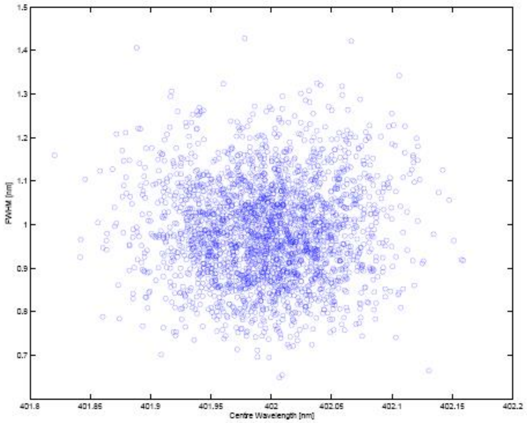
Correlation: Type A and Type B methods

Correlation occurs when there is something in common between the different quantities, or between the measured values being combined.

Type A: From the data

Type B: From knowledge





$$E_i = E_{\text{True}} + S + R_i$$

This is where the correlation comes from!

Systematic Effects!

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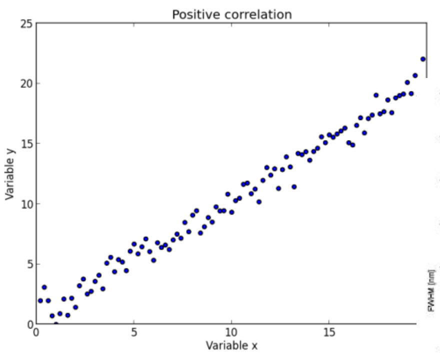
Correlation: Type A and Type B methods

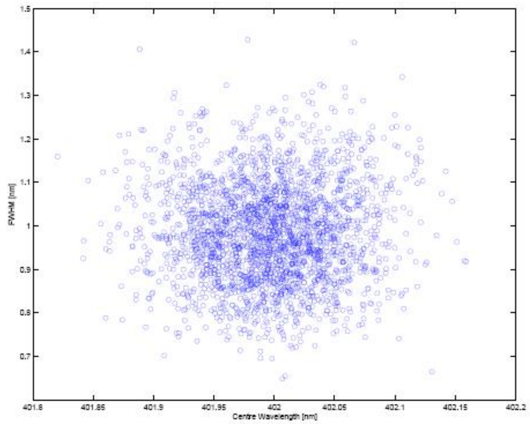
Correlation occurs when there is something in common between the different quantities, or between the measured values being combined.

Type A: From the data

$$r(x,y)=\frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{X_i-\bar{X}}{s_X}\right)\left(\frac{Y_i-\bar{Y}}{s_Y}\right)$$

$$u(x,y)=u(x)u(y)r(x,y)$$





$$u_c^2(y)=\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i)+\boxed{2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i,x_j)}$$

Correlation?

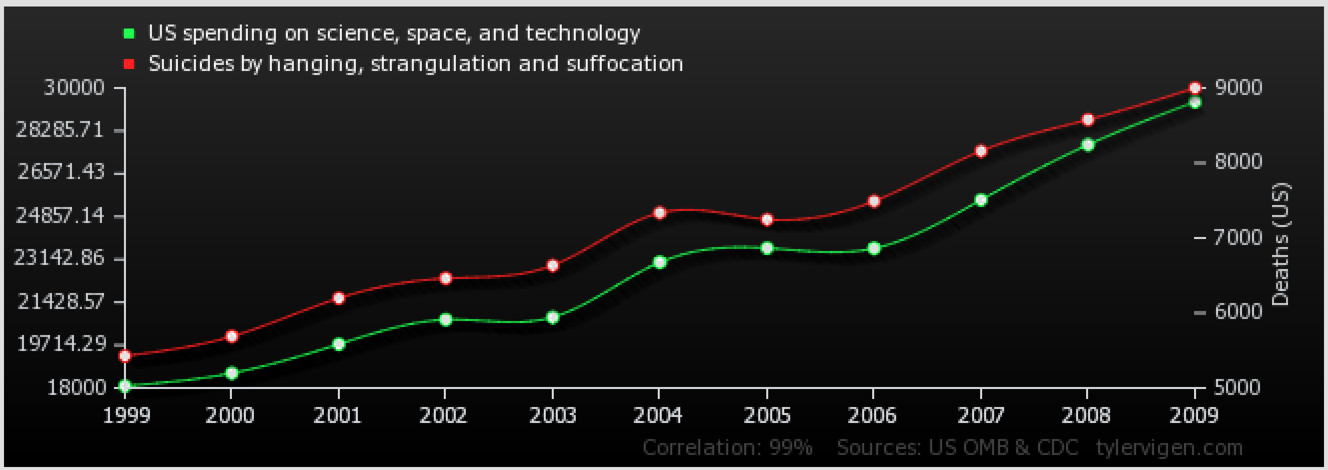
http://www.tylervigen.com/

Spurious correlations

US spending on science, space, and technology

correlates with

Suicides by hanging, strangulation and suffocation



	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009
US spending on science, space, and technology *Millions of todays dollars (US OMB)*	18,079	18,594	19,753	20,734	20,831	23,029	23,597	23,584	25,525	27,731	29,449
Suicides by hanging, strangulation and suffocation *Deaths (US) (CDC)*	5,427	5,688	6,198	6,462	6,635	7,336	7,248	7,491	8,161	8,578	9,000

Correlation: 0.992082

Permalink - Mark as interesting (5,147) - Not interesting (2,370)

Correlation: Type A and Type B methods

Correlation occurs when there is something in common between the different parameters, or between the measurements being combined.

Calculate covariance

Remove covariance

Type B: From knowledge

$$E_i = E_{\text{True}} + S + R_i$$

This is where the correlation comes from!

Systematic Effects!

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Averaging partially correlated data

$$E_i = E_{\mathrm{True}} + S + R_i$$

$$E_{\mathrm{M}} = \frac{E_1 + E_2 + E_3}{3}$$

$$E_{\mathrm{M}} = \frac{3E_{\mathrm{True}}}{3} + \frac{3S}{3} + \frac{R_1 + R_2 + R_3}{3}$$

$$u^2\left(E_{\mathrm{M}}\right) = u^2\left(S\right) + \left(\frac{u\left(R_i\right)}{\sqrt{3}}\right)^2$$

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Averaging Partially correlated data

$$u_c^2(y) = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i, x_j)$$

$$u(E_i) = u^2(S) + u^2(R_i)$$

$$u(E_i, E_j) = u^2(S); \qquad i \neq j$$

$$E_i = E_{\text{True}} + S + R_i$$

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Averaging partially correlated data

$$u_c^2(y) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \right)^2 u^2(x_i) + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} u(x_i, x_j)$$

$$u(E_1, E_2), u(E_1, E_3), u(E_1, E_4), \ldots$$

$$u(E_2, E_3), u(E_2, E_4), \ldots$$

$$u(E_3, E_4), \ldots$$

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Averaging partially correlated data

$$E_i = E_{\mathrm{True}} + S + R_i$$

$$E_{\mathrm{M}} = \frac{E_1 + E_2 + E_3}{3}$$

$$\frac{\partial E_{\mathrm{M}}}{\partial E_1} = \frac{\partial E_{\mathrm{M}}}{\partial E_2} = \frac{\partial E_{\mathrm{M}}}{\partial E_3} = \frac{1}{3}$$

$$\boxed{u(E_i) = u^2(S) + u^2(R_i)}$$

$$\boxed{u(E_i, E_j) = u^2(S); \qquad i \neq j}$$

$$u_{\mathrm{c}}^2(y) = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i, x_j)$$

$$u^2(E_{\mathrm{M}}) = 3\left(\frac{1}{3}\right)^2 u^2(S) + 3\left(\frac{1}{3}\right)^2 u^2(R_i) + 3\times 2\times\left(\frac{1}{3}\right)^2 u^2(S)$$

$$u^2(E_{\mathrm{M}}) = (3+6)\left(\frac{1}{3}\right)^2 u^2(S) + 3\left(\frac{1}{3}\right)^2 u^2(R_i)$$

$$\boxed{u^2(E_{\mathrm{M}}) = u^2(S) + \left(\frac{u(R_i)}{\sqrt{3}}\right)^2}$$

Systematic and random effects: Lamp measured 5 times (continued)

Systematic effects	Random effects
Reference calibration	Noise
Alignment	Lamp current fluctuation
Lamp current setting	
Temperature sensitivities	

$$S, u(S)$$

$$R_i, u(R_i)$$

$$u^2(E_{\mathrm{M}}) = u^2(S) + \left(\frac{u(R_i)}{\sqrt{n}}\right)^2$$

$$E_i = E_{\mathrm{True}} + S + R_i$$

$$u^2(E_i) = u^2(S) + u^2(R_i)$$

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Averaging partially correlated data

$$u_c^2(y) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \right)^2 u^2(x_i) + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} u(x_i, x_j)$$

$$u(E_1, E_2), u(E_1, E_3), u(E_1, E_4), \ldots$$

$$u(E_2, E_3), u(E_2, E_4), \ldots$$

$$u(E_3, E_4), \ldots$$

Covariance Matrix

$$
\begin{array}{cc}
 & \begin{array}{cccc} 1 & 2 & \cdots & n \end{array} \\
\boldsymbol{U}_E = \begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array} &
\begin{bmatrix}
u^2(E_1) & u(E_1,E_2) & \cdots & u(E_1,E_n) \\
u(E_2,E_1) & u^2(E_2) & \cdots & u(E_2,E_n) \\
\vdots & \vdots & \ddots & \vdots \\
u(E_n,E_1) & u(E_n,E_2) & \cdots & u^2(E_n)
\end{bmatrix}
\end{array}
$$

$$
u_c^2(y) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \right)^2 u^2(x_i) + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} u(x_i, x_j)
$$

$$
u(E_1,E_2), u(E_1,E_3), u(E_1,E_4), \ldots
$$

$$
u(E_2,E_3), u(E_2,E_4), \ldots
$$

$$
u(E_3,E_4), \ldots
$$

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Covariance Matrix Version

$$u_c^2(y) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \right)^2 u^2(x_i) + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} u(x_i, x_j)$$

$$u^2(y) = \boldsymbol{C}_y \boldsymbol{U}_x \boldsymbol{C}_y^{\top} \qquad \boldsymbol{C}_y = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{pmatrix}$$

$$\boldsymbol{U}_E = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots & n \end{array} \\ \begin{bmatrix} u^2(E_1) & u(E_1, E_2) & \cdots & u(E_1, E_n) \\ u(E_2, E_1) & u^2(E_2) & \cdots & u(E_2, E_n) \\ \vdots & \vdots & \ddots & \vdots \\ u(E_n, E_1) & u(E_n, E_2) & \cdots & u^2(E_n) \end{bmatrix} \end{array}$$

Covariance matrix with absolute variance (squared uncertainty) and covariance

Covariance matrix

$$\boldsymbol{U}_E = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & \cdots & n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array} & \begin{bmatrix} u^2(E_1) & u(E_1,E_2) & \cdots & u(E_1,E_n) \\ u(E_2,E_1) & u^2(E_2) & \cdots & u(E_2,E_n) \\ \vdots & \vdots & \ddots & \vdots \\ u(E_n,E_1) & u(E_n,E_2) & \cdots & u^2(E_n) \end{bmatrix} \end{array}$$

$$E(\lambda_i) = E_{\mathrm{T}}(\lambda_i)(1+S)(1+R_i) + \tilde{s} + \tilde{r}_i$$

$$\tilde{U}_{E,ij} = \begin{cases} E_i^2\left[u^2(S)+u^2(R_i)\right]+u^2(\tilde{s})+u^2(\tilde{r}_i) & (i=j) \\ E_iE_ju^2(S)+u^2(\tilde{s}) & (i \neq j). \end{cases}$$

Negative sensitivity coefficients

$$y = A - B \qquad \frac{\partial y}{\partial A} = 1 \quad \frac{\partial y}{\partial B} = -1$$

$$u^2(y) = u^2(A) + (-1)^2 u^2(B)$$

$$u^2(y) = u^2(A) + u^2(B)$$

E.g. Signal = Light - Dark

Negative sensitivity coefficients

$$y = A - B \qquad \frac{\partial y}{\partial A} = 1 \quad \frac{\partial y}{\partial B} = -1$$

$$u^2(y) = u^2(A) + u^2(B) + 2(1)(-1)u(A, B)$$

$$u^2(y) = u^2(A) + u^2(B) - 2u(A, B)$$

Correlation reduces uncertainty

E.g. Signal = Light - Dark

Positive sensitivity coefficients

$$y = A + B \qquad \frac{\partial y}{\partial A} = 1 \quad \frac{\partial y}{\partial B} = 1$$

$$u^2(y) = u^2(A) + u^2(B) + 2(1)(1)u(A,B)$$

$$u^2(y) = u^2(A) + u^2(B) + 2u(A,B)$$

Correlation increases uncertainty

E.g. Distance = measured + thickness

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But I don’t know what the covariance is!

- Think of the worst-case scenario

Does correlation increase or decrease the uncertainty?

Treat as “all random” and “all systematic”

At the end of this module, you should understand

- The Law of Propagation of Uncertainties
  - Why the sensitivity coefficient is central to the law
  - How to determine the sensitivity coefficients
  - What ‘adding in quadrature’ really means

- How many readings you should average
  - What the uncertainty associated with an average is
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  - When you’ve taken too many readings

- What correlation is
  - What the uncertainties are for averaging partially correlated data
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CONCLUSIONS AND SUMMARY

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At the end of this module, you should understand

- The Law of Propagation of Uncertainties
    - Why the sensitivity coefficient is central to the law
    - How to determine the sensitivity coefficients
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- How many readings you should average
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Sensitivity Coefficients

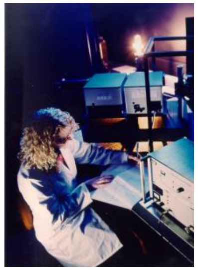
$$u_c^2(y) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \right)^2 u^2(x_i) + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} u(x_i, x_j)$$

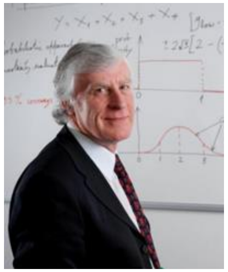
Experimentally

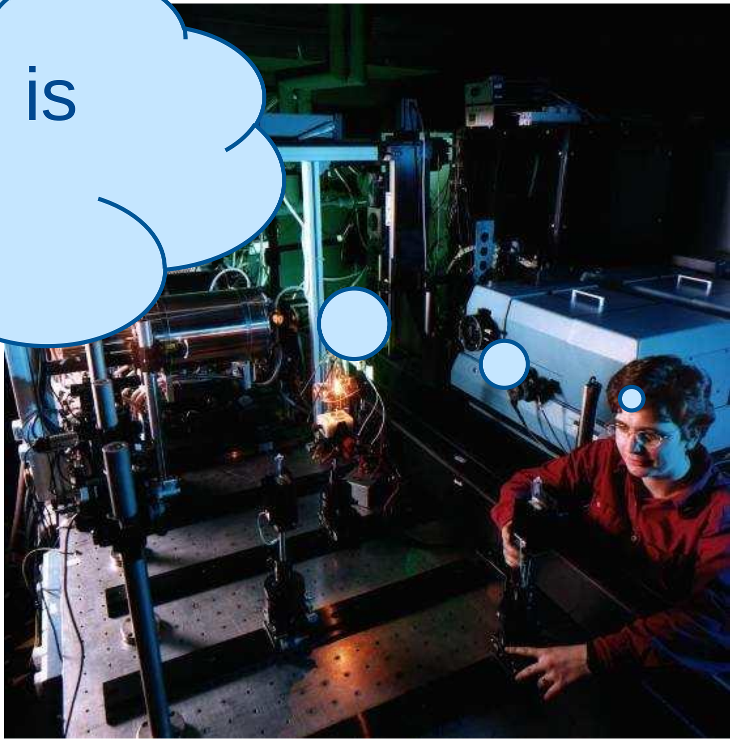
Mathematically

Numerically

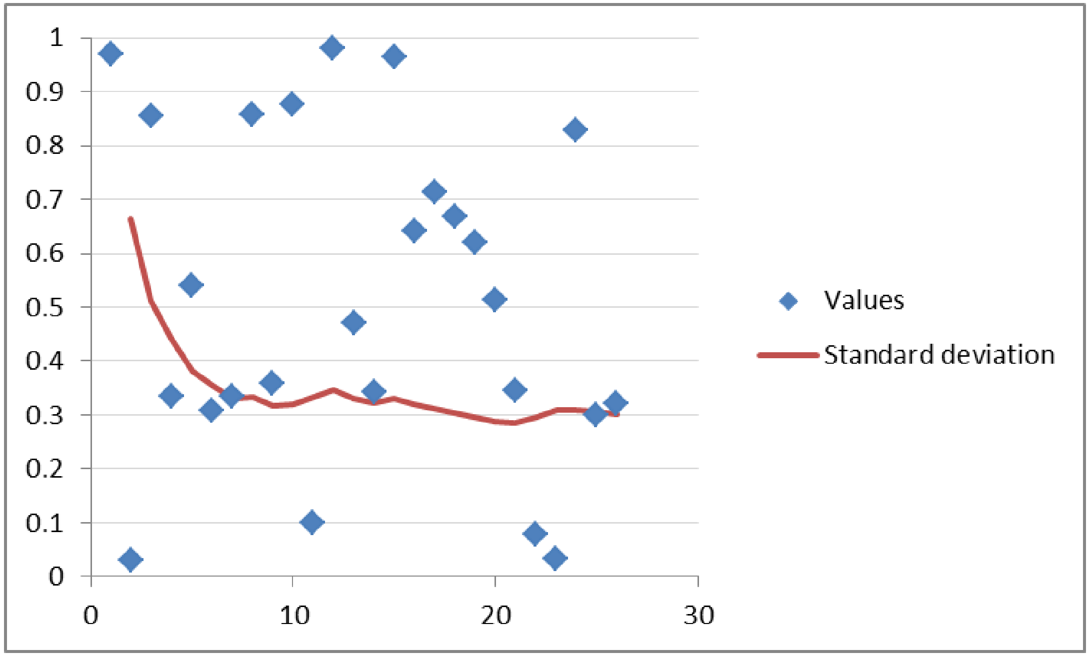
How sensitive is my result to this?







Taking enough readings, but not too many

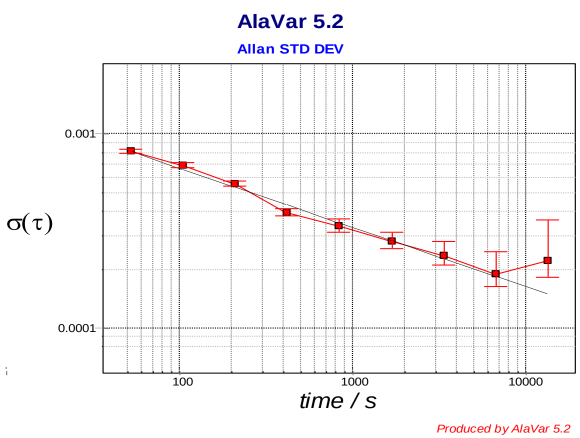


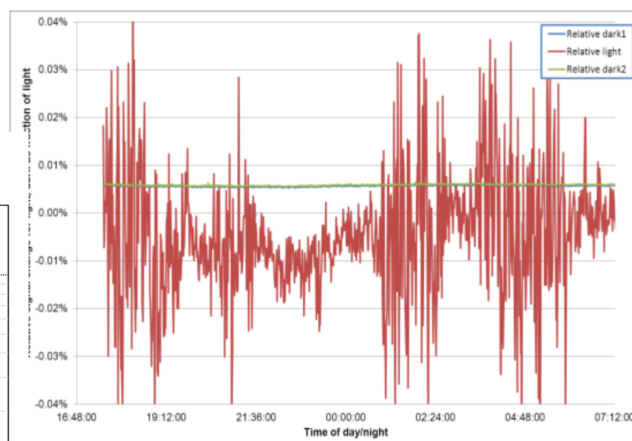
Better estimate of uncertainty

More to average

$$u\left(E_{\mathrm{M}}\right)=\left(\frac{u\left(E_i\right)}{\sqrt{n}}\right)$$

Allan Deviation





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Correlation

Correlation occurs when there is something in common between the different parameters, or between the measurements being combined.

Calculate covariance

Remove covariance

Type B: From knowledge

$$E_i = E_{\text{True}} + S + R_i$$

This is where the correlation comes from!

Systematic Effects!

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